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Quantum-like picture for intrinsic, classical, arrival distributions

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Abstract
We introduce a marginal, quantum-like picture for the arrival of classical quantities in which the representation vectors are the quantities that evolve and probability densities remain static. The representation functions can be seen as probe functions which are the evolution of delta functions with support on a curve in phase space, the time fronts. This procedure provides a classical analog as well as a clear physical interpretation of the 'time eigenstates' used in quantum systems.

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1. Introduction

There is an interest in finding a way of dealing with time in many areas of physics (see, for example, [1] and references therein). For quantum systems, an early statement about the need for a way of treating the time variable was made by Pauli [2–4] who pointed out that there is no time operator canonically conjugate to a semi-bounded Hamiltonian. Allcock [5–7] has also given a proof of the nonexistence of a time-of-arrival operator in the usual Hilbert space. However, there are various analyses and criticisms of Pauli's assertion [8, 9], leading to proposals for the calculation of intrinsic or operational arrival-time distributions, for time operators [9–33], and for conjugate pairs of the Hamiltonian [34]. The existing literature on this subject is vast; a few references are listed at the end of this paper.

Some authors have put aside the time variable that appears in the equations of motion and have taken a component of the system as the clock. Since those clocks might not be linearly related to usual clocks, only in limiting cases can one recover the usual equations of motion [12, 35–37]. There is also the treatment which includes an external system coupled to the one of interest in order to model an experimental arrangement [23].

For a single classical particle it is possible to perform a canonical transformation to an energy–time description of the dynamics [19, 20, 38, 39]. However, the variable time continues to appear as a parameter, and an additional time parameter appears. Further, an energy–time description for classical probability densities has not yet been developed.

This paper serves two purposes. One is to introduce a picture of classical dynamics of point particles which assigns a time value to points in phase space. This is intended to provide an alternative means of describing the dynamics of probability densities and calculating arrival distributions. Second, this treatment is modeled keeping in mind the concepts used in quantum mechanics, and is of great help in the elucidation of time eigenstates and quantum arrival distributions.

The picture that we introduce is intended to handle classical distributions and is the classical analog of a picture of quantum dynamics which was introduced in [40]. That picture is able to address arrival-time-related questions and puts forward a common language for the analysis of classical and quantum systems. We show how to calculate arrival-time distributions with classical probability densities, and we find that some of the difficulties found in quantum systems are also found in classical ones. These results are an intermediate step on the path to an energy–time representation for the dynamics of classical densities, as has already been developed for quantum wave packets [41].

The results presented in this paper are inspired by previous works on quantum time and arrival-time distributions, specially by the quantum 'time eigenstates'. A well-known 'time eigenstate' for quantum free motion is Kijowski's function, which, in the momentum representation, is given by [32]

$$\langle p|T,\alpha\rangle = \Theta(\alpha p)\,\mathrm{e}^{\mathrm{i}p^2T/2m\hbar}\sqrt{\frac{|p|}{m\hbar}}, \tag{1}$$

where $\Theta(\alpha p)$ is the Heaviside step function and $\alpha = \pm 1$ for right and left movers, respectively. The squared magnitude of this state corresponds to the flux that arrives at $q = 0$ and at time T. This state was obtained from a modification of the Hamiltonian of the free particle so that it could have positive and negative energy eigenvalues. As was noted by Baute [15], Kijowski's time eigenstates for free motion [22] can be seen as the backward evolution of the initial state $\Theta(\alpha p)\sqrt{|p|/m\hbar}$, which corresponds to an initial probability density $\Theta(\alpha p)|p|/m\hbar$. The arrival-time distribution is the squared modulus of the inner product between this state and a given initial wave packet, $|\psi\rangle$, for which one is interested in determining the arrival-time distribution:

$$\Pi(T,\alpha) = |\langle T,\alpha|\psi\rangle|^2. \tag{2}$$

Baute *et al* have also generalized this to arbitrary potentials by replacing the factor $\mathrm{e}^{\mathrm{i}p^2T/2m\hbar}$ with $\mathrm{e}^{\mathrm{i}T\hat{H}/\hbar}$, where $\hat{H}$ is the Hamiltonian operator [15]. On the other hand, the classical analog of the quantum arrival-time distribution is the modulus of the flux of particles that arrive at X *from one side* at a given time [25]:

$$\langle J_\pm(X)\rangle = \int_0^\infty \mathrm{d}p \int_{-\infty}^\infty \mathrm{d}q\ \rho(\pm p, q; t)\frac{|p|}{m}\delta(q - X). \tag{3}$$

In this paper, we use 'probe functions' (the analog of the quantum time eigenstates) which are propagated backward in time according to the Liouville equation, as we usually do with any other dynamical variable. These probe functions sample initial probability densities and then predict arrival distributions. They are well-localized functions in phase space. The analysis of the classical problem clarifies what the states of Kijowski and others do: they pick the part of the initial probability density that will arrive at $q = X$ at the arrival time T. This is a new picture of classical mechanics in which the time concept is well defined; the probe functions (representation functions) evolve in time and probability densities do not. The probe

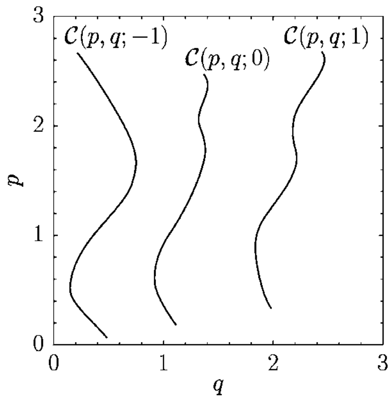

Figure 1. The time grid. An arbitrary curve $\mathcal{C}(p, q; 0)$ is taken as the set of zero-time points in phase space and then it is propagated backward and forward, generating a time grid in phase space. Dimensionless units.

functions are solutions of the time-dependent evolution equation, and they are better suited for answering time-related questions.

In the following sections, we propose a convenient method of studying the dynamics in phase space when one is interested in arrival distributions: by means of marginal probe functions. We define the probe functions and explore their properties. At the end, there are some concluding remarks.

2. A picture of classical dynamics

In this section, we consider one-dimensional classical probability densities evolving on a non-constant potential function and we introduce classical 'probe functions' with support on 'time-front' curves in phase space. These functions are similar to the 'time eigenstates' used in quantum systems, as described by Aharonov [10], Kijowski [32], Muga [1], Skulimowski [18] and Torres-Vega [40]. The classical analysis in this paper is of great help in the elucidation of the physical meaning of quantum time eigentates.

We start by noting that the time variable that appears in Hamilton's equations of motion (we will denote it by T) is meant to be associated with a single particle, but the time variable that appears in Liouville's equation of motion (which we will denote by t) is meant to be associated with an *ensemble* of particles.

Now, we define a time grid in phase space as follows. We take any curve $\mathcal{C}(p, q)$ in phase space with the properties that it covers the values of momentum p of interest and does not intersect with itself (see figure 1). When the points of this curve are evolved by a time T, according to Hamilton's equations, we obtain a new curve $\mathcal{C}(p, q; T)$ in phase space. Next, we assign to the points of $\mathcal{C}(p, q)$ the time value $T = 0$ and to the points in the curve $\mathcal{C}(p, q; T)$ the time value T. Then, we will say that the points that belong to $\mathcal{C}(p, q; T)$ have a time T, forming a time grid in phase space.

For arrival-time distributions, we should use as the initial curve $\mathcal{C}(p, q)$ the line of constant $q = X$. When this line is backward and forward propagated, the resulting curves will form the time grid. For free particles, the time grid is given by

$$m(X - q) + pT = 0, \tag{4}$$

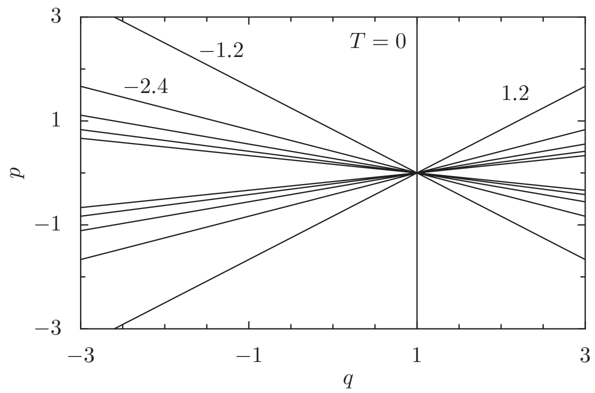

Figure 2. Time grid for free particles and for arrival at $q = 1$. Time fronts with some of them labeled with their corresponding values of time. Dimensionless units.

and a plot of them is shown in figure 2. These curves look the same independently of the value of X, all centered on $q = X$, in accordance with the translational symmetry of the system.

We will use a framework in which there are probe functions that evolve in phase space, whereas the probability densities are static. We deduce some of their properties, and use them in the determination of arrival quantities. Classical densities have also been used for the analysis of tunneling, and that has been of help in understanding the quantum phenomena [42, 43].

At a given arrival time T, only a slice of the density $\rho(\Gamma; T)$ steps at a given point $q = X$ and then the amount of probability at that point, the presence distribution, is given by

$$\Pi(T; X) = \Pi^{+}(T; X) + \Pi^{-}(T; X), \tag{5}$$

where we have separated the contributions from left and right movers:

$$\Pi^{-}(T; X) = \int_{-\infty}^{0-} \mathrm{d}p\, \rho(p, q = X; T), \tag{6}$$

and

$$\Pi^{+}(T; X) = \int_{0}^{\infty} \mathrm{d}p\, \rho(p, q = X; T). \tag{7}$$

The point $p = 0$ is excluded from the left mover contribution, in order to avoid taking this value into account twice. In the rest of the paper, we sometimes use the notation $\int_{-\infty}^{\infty} \mathrm{d}p\, \Theta(\pm p)$ to include both cases—left and right movers—but it should be understood that the value $p = 0$ is to be excluded from the left mover integrals.

Alternatively, we determine arrival distributions from an initial probability density $\rho(p, q; T = 0)$ by finding the slice that will be aligned at the point $q = X$ at time T. The presence distribution can be rewritten as

$$\begin{aligned}\Pi(T; X) &= \int_{-\infty}^{\infty} \mathrm{d}p\, \rho(p, X; T) \\ &= \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p\, \delta(q - X)\, \mathrm{e}^{-T\hat{\mathcal{L}}(p,q)} \rho(p, q)\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p\, \rho(p,q) \mathrm{e}^{T\hat{\mathcal{L}}(p,q)} \delta(q - X) \\
&\equiv \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p\, \rho(p,q) \nu(\Gamma; T; X), \qquad (8)
\end{aligned}
$$

where

$$\hat{\mathcal{L}}(\Gamma) \equiv \frac{p}{m}\frac{\partial}{\partial q} + F(q)\frac{\partial}{\partial p} \qquad (9)$$

is the Liouville operator, m is the mass of the particles, $F(q)$ is the force that is experienced by the particles and

$$
\begin{aligned}
\nu(\Gamma; T; X) &\equiv \mathrm{e}^{T\hat{\mathcal{L}}(p,q)} \delta(q - X) \\
&= \delta(q(T) - X) \qquad (10)
\end{aligned}
$$

is a probe function that samples the initial probability density for arrival at time T. Here, $(p(T), q(T))$ is the propagated phase-space point that starts at (p, q). Thus, this function selects the points (p, q) that, after evolution by a time T, arrive at $q = X$. This is equivalent to the set of points obtained by the backward evolution of the line $q = X$.

We can also ask for the arrival of some quantity $\mathcal{O}(\Gamma)$. The appropriate probe functions are defined as

$$
\begin{aligned}
\nu_{\mathcal{O}}^{\pm}(\Gamma; T; X) &\equiv \Theta(\pm p)\, \mathrm{e}^{T\hat{\mathcal{L}}(p,q)} \delta(q - X)\mathcal{O}(p,q) \\
&= \Theta(\pm p)\delta(q(T) - X)\mathcal{O}(p(T), q(T)), \qquad (11)
\end{aligned}
$$

for right and left movers (positive and negative momentum). $\mathcal{O}(\Gamma) = 1$ for presence distributions (in what follows, if $\mathcal{O}(\Gamma) = 1$ we omit the subscript $\mathcal{O}$) and $\mathcal{O}(\Gamma) = |p|/m$ for arrival-time distributions. For instance, for the free particle, the arrival probe functions are,

$$\nu_{|p|/m}^{\pm}(\Gamma; T, X) = \Theta(\pm p)\frac{|p|}{m}\delta\left(q + \frac{p}{m}T - X\right), \qquad (12)$$

a delta function with the time-front curve (4) as a support.

The probe function $\nu(\Gamma; T; X)$ is the evolution of a delta function with support on the line at $q = X$ and parallel to the momentum axes, taking it as the origin of time. In quantum systems, the state $\sqrt{|p|/m}\, \mathrm{e}^{\mathrm{i}t\hat{H}/\hbar + \mathrm{i}pq/\hbar}$, in momentum space, is commonly used as a time eigenstate for arrival at $q = 0$. The squared modulus of the factor $\mathrm{e}^{\mathrm{i}t\hat{H}/\hbar + \mathrm{i}pq/\hbar}$, at $t = 0$, is constant in p, but a delta function, $\delta(q)$, in coordinate space, i.e. the same behavior as the probe function introduced in this paper, and, therefore, it is its classical analog. Quantum time eigenstates for free particles can be found in [21], for instance.

The time-front curves $\mathcal{C}(p, q; T)$ can become quite complicated, especially when there is recurrence in the system, when particles cross the line $q = X$ several times. In order to provide an idea of how these curves behave, figure 3 contains a plot of the support of one of the propagated probe functions for arrival at $q = 2.5$ from left and right (for numerical calculations, we use dimensionless quantities such that $\hbar = m = 1$). The propagation has been carried out for a classical system with a symmetrical Eckart potential $V(x) = V_0/\cosh^2(x)$ [44]. This type of model has been quite used in the analysis of quantum tunneling [44, 45], besides the square potential barrier [42, 43, 46].

If we put a density of constant height on the support of $\nu_{\mathcal{O}}(\Gamma; T; X)$, for example on the curve of figure 3, and integrate over the momentum, we obtain the densities presented in figure 4. These would be the classical analogs of the quantum presence probe functions in coordinate space.

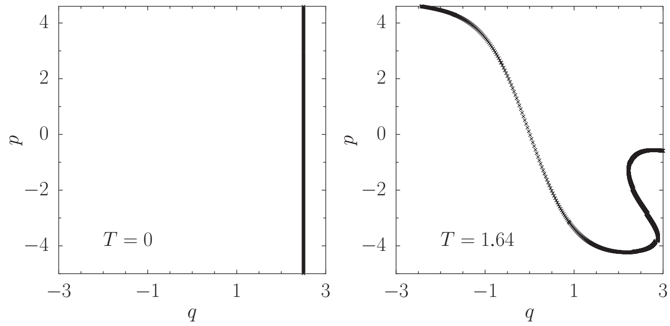

Figure 3. Support of the probe function $\nu(\Gamma; T; X)$, for arrival at $X = 2.5$ with a symmetric Eckart potential of strength $V_0 = 10$. We show the curve for $T = 0$, and 1.64. The overlap between the probe function and some initial probability density gives the presence distribution at $X = 2.5$ at time T, equation (8). Dimensionless units.

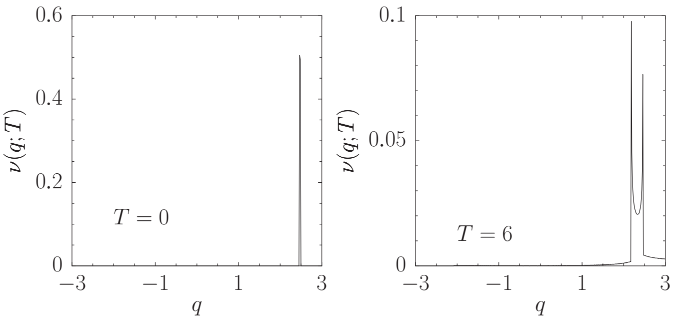

Figure 4. Projections over q of a constant density on the support of $\nu_{\mathcal{O}}(\Gamma; T; X)$, for the Eckart potential $V(x) = \mathrm{e}^x/(2\cosh(x)) + B/(4A\cosh^2(x))$, with $B/A = 4$ and different times. Dimensionless units.

3. Properties of probe functions

In this section, we introduce some classical equivalents of the properties of quantum time eigenstates.

3.1. Covariance

When the probe function at time T is propagated for a time τ, what we get is the probe function for the new time $T + \tau$:

$$\mathrm{e}^{\tau\hat{\mathcal{L}}(\Gamma)}\nu_{\mathcal{O}}(\Gamma; T; X) = \mathrm{e}^{\tau\hat{\mathcal{L}}(\Gamma)}\,\mathrm{e}^{T\hat{\mathcal{L}}(\Gamma)}\delta(q - X)\mathcal{O}(p, q)$$
$$= \nu_{\mathcal{O}}(\Gamma; T + \tau; X), \tag{13}$$

$$\mathrm{e}^{\tau\hat{\mathcal{L}}(\Gamma)}\nu_{\mathcal{O}}^{\pm}(\Gamma; T; X) = \nu_{\mathcal{O}}^{\pm}(\Gamma; T + \tau; X). \tag{14}$$

This is a characteristic of a clock variable, and is the equivalent of the so-called covariance of quantum time eigenstates.

The time grid is obtained precisely in this way; we take a curve which will be the zero-time time front and propagate it forward and backward generating the time grid (see figures 1, 2, 3 and 6).

3.2. Orthogonality

As time goes on, the front curves evolve extending over the phase space. Then, it is expected that the probe functions for different times and the same X overlap at some point. This can be appreciated in figures 2 and 3. From equations (13) and (14), we find that

$$\begin{aligned}\int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}^{\pm}(\Gamma; T'; X)\nu_{\mathcal{O}}^{\pm}(\Gamma; T; X) \\ &= \int \mathrm{d}\Gamma\big[\mathrm{e}^{T'\hat{\mathcal{L}}(\Gamma)}\nu_{\mathcal{O}}^{\pm}(\Gamma; 0; X)\big]\nu_{\mathcal{O}}^{\pm}(\Gamma; T; X) \\ &= \int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}^{\pm}(\Gamma; 0; X)\, \mathrm{e}^{-T'\hat{\mathcal{L}}(\Gamma)}\nu_{\mathcal{O}}^{\pm}(\Gamma; T; X) + \mathrm{b.t.} \\ &= \int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}^{\pm}(\Gamma; 0; X)\nu_{\mathcal{O}}^{\pm}(\Gamma; T - T'; X) + \mathrm{b.t.},\end{aligned} \tag{15}$$

also

$$\begin{aligned}\int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}^{+}(\Gamma; T'; X)\nu_{\mathcal{O}}^{-}(\Gamma; T; X) \\ &= \int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}^{+}(\Gamma; 0; X)\nu_{\mathcal{O}}^{-}(\Gamma; T - T'; X) + \mathrm{b.t.},\end{aligned} \tag{16}$$

and

$$\begin{aligned}\int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}(\Gamma; T'; X)\nu_{\mathcal{O}}(\Gamma; T; X) \\ &= \int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}(\Gamma; 0; X)\nu_{\mathcal{O}}(\Gamma; T - T'; X) + \mathrm{b.t.},\end{aligned} \tag{17}$$

where b.t. indicates the boundary terms originating in the integration by parts. For presence functions and when $\mathcal{O}(\Gamma)$ is zero or periodic at the boundaries, the boundary terms vanish. These are the overlaps between the probe functions at time $T - T'$ with the line $q = X$ (see figures 2 and 3).

However, they do not overlap for different X and the same arrival time:

$$\begin{aligned}\int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}(\Gamma; T; X')\nu_{\mathcal{O}}(\Gamma; T; X) \\ &= \int \mathrm{d}\Gamma[\mathrm{e}^{T\hat{\mathcal{L}}(\Gamma)}\delta(q - X')\mathcal{O}(\Gamma)]\, \mathrm{e}^{T\hat{\mathcal{L}}(\Gamma)}\delta(q - X)\mathcal{O}(\Gamma) \\ &= \int \mathrm{d}\Gamma\, \delta(q - X')\mathcal{O}(\Gamma)\, \mathrm{e}^{-T\hat{\mathcal{L}}(\Gamma)}\, \mathrm{e}^{T\hat{\mathcal{L}}(\Gamma)}\delta(q - X)\mathcal{O}(\Gamma) + \mathrm{b.t.} \\ &= \int \mathrm{d}\Gamma\, \delta(q - X')\mathcal{O}(\Gamma)\delta(q - X)\mathcal{O}(\Gamma) + \mathrm{b.t.} \\ &= \int \mathrm{d}p\, \mathcal{O}^2(p, X)\delta(X - X') + \mathrm{b.t.} \\ &= \delta(X - X') \int \mathrm{d}p\, \mathcal{O}^2(p, X) + \mathrm{b.t.}\end{aligned} \tag{18}$$

This is illustrated for the Eckart potential in figure 5.

Thus, in a manner similar to that of quantum probe states, these probe functions are non-normalizable and overlap for the same X and different T; however, they do not overlap for different X and the same T [40].

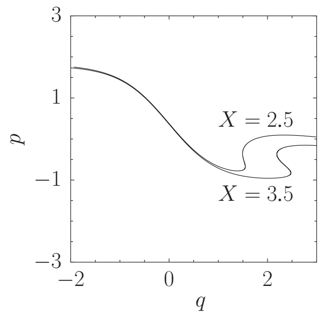

Figure 5. Orthogonality of probe functions for different X and equal T. Here, $T = 6$ and the potential is the Eckart potential $V(x) = e^x/(2\cosh(x)) + B/(4A\cosh^2(x))$, with $B/A = 4$. Dimensionless units.

3.3. Averages

The distribution $\Pi_{\mathcal{O}}(X, T)$ and probe function $\nu_{\mathcal{O}}(\Gamma; X, T)$ depend on X and T, so we can integrate over these variables. The time-dependent average of $\mathcal{O}(\Gamma)$ can be calculated with $\Pi_{\mathcal{O}}(T; X)$

$$\begin{aligned}\int_{-\infty}^{\infty} dX\, \Pi_{\mathcal{O}}(T; X) &= \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dp \mathcal{O}(p, X)\rho(p, X; T)\\ &= \langle \mathcal{O}(T)\rangle.\end{aligned} \tag{19}$$

There is an equivalence between a time and phase-space integration:

$$\begin{aligned}\int_{-\infty}^{\infty} dT\, \Pi_{\mathcal{O}}(T; X) &= \int_{-\infty}^{\infty} dT \int d\Gamma\, \rho(\Gamma)\mathcal{O}(\Gamma; T)\delta(q(T) - X)\\ &= \int_{-\infty}^{\infty} dT \int d\Gamma\, \rho(\Gamma)\mathcal{O}(\Gamma; T)\frac{\delta(T + T_X)}{|\dot{q}(T)|}\\ &= \int d\Gamma\, \rho(\Gamma)\frac{\mathcal{O}(\Gamma; -T_X)}{|\dot{q}(-T_X)|}\\ &= \int d\Gamma\, \rho(\Gamma)\frac{m}{|p(-T_X)|}\mathcal{O}(\Gamma; -T_X)\\ &= \int d\Gamma\, \frac{m}{|p|}\mathcal{O}(\Gamma)\rho(\Gamma; -T_X).\end{aligned} \tag{20}$$

The corresponding calculation for probe functions is

$$\begin{aligned}\int_{-\infty}^{\infty} dT\, \nu_{\mathcal{O}}(\Gamma; T; X) &= \int_{-\infty}^{\infty} dT\, \mathcal{O}(\Gamma; T)\delta(q(T) - X)\\ &= \int_{-\infty}^{\infty} dT\, \mathcal{O}(\Gamma; T)\frac{\delta(T + T_X)}{|\dot{q}(T)|}\\ &= \int_{-\infty}^{\infty} dT\, \mathcal{O}(\Gamma; T) m \frac{\delta(T + T_X)}{|p(T)|}\\ &= m\frac{\mathcal{O}(\Gamma; -T_X)}{|p(-T_X)|}.\end{aligned} \tag{21}$$

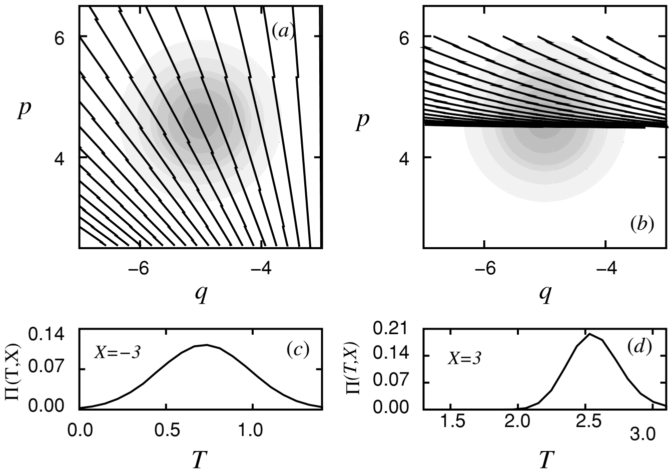

Figure 6. Presence distribution for particles experiencing the Eckart potential. (a) A density plot of an initial Gaussian probability density $\rho(\Gamma)$ and of $\nu(\Gamma; T; X)$ for $X = 3$ and several times, and (b) for $X = -3$. The corresponding $\Pi(T; X)$ are plotted in (c) and (d). Dimensionless units.

Another result involving the integration over X is that the observable $\mathcal{O}$ at time T is obtained by integrating $\nu_{\mathcal{O}}(\Gamma; T; X)$ over X:

$$\int_{-\infty}^{\infty} \mathrm{d}X\, \nu_{\mathcal{O}}(\Gamma; T; X) = \int_{-\infty}^{\infty} \mathrm{d}X\, \delta(q(T) - X)\mathcal{O}(p(T), q(T)) \\ = \mathcal{O}(\Gamma; T). \tag{22}$$

3.4. Projectors

For the purpose of knowing arrival distributions, one only needs to know the amount arriving at time T, $\Pi_{\mathcal{O}}(T; X)$, and, perhaps, the region from which it is coming, i.e. $\nu(\Gamma; T; X)$.

The use of projection operators $\hat{P} = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a wavefunction, is common in quantum mechanics, and one of their basic properties is that $\hat{P}^2 = \hat{P}$. We can form a kind of 'classical projection operator' by making use of probe functions.

The projector $\hat{P}_{\mathcal{O}}$, acting on ρ, is defined as

$$\hat{P}_{\mathcal{O}}\rho(\Gamma) \equiv \nu(\Gamma; T; X) \int \mathrm{d}\Gamma'\, \nu_{\mathcal{O}}(\Gamma'; T; X)\rho(\Gamma') \\ = \nu(\Gamma; T; X)\Pi_{\mathcal{O}}(T, X). \tag{23}$$

Then, $\hat{P}_{\mathcal{O}}\rho(\Gamma)$ is the ν representation of $\mathcal{O}(\Gamma)\rho(\Gamma)$, good for determining arrival probabilities.

$\hat{P}_{\mathcal{O}}\rho(\Gamma)$ is the overlap between $\nu_{\mathcal{O}}(\Gamma; T; X)$ and $\rho(\Gamma)$ times a delta function which supports the time-front curves. In figure 6, we have plotted an initial Gaussian probability density, $\rho(\Gamma) = \exp(-(q - q_0)^2 - (p - p_0)^2)$ centered at $(q_0, p_0) = (-5, 4.6)$, together with the support of $\nu(\Gamma; T, X)$ for arrival at $X = -3$ and 3. There is also a plot of the corresponding presence distribution $\Pi(T; X)$. The time-front curves in this case are no longer straight indicating that the curve hits the density at a later time compared with the free particle.

With this projection, the details of $\rho(\Gamma)$ are lost and it cannot be recovered from $\Pi_{\mathcal{O}}(T; X)$ and $\nu(\Gamma; T; X)$.

A second application of $\hat{P}$ results in

$$\hat{P}^2_{\mathcal{O}}\rho(\Gamma) = \nu(\Gamma; T; X)\left[\int \mathrm{d}\Gamma'\, \nu_{\mathcal{O}}(\Gamma'; T; X)\nu(\Gamma'; T; X)\right] \times \int \mathrm{d}\Gamma''\nu_{\mathcal{O}}(\Gamma''; T; X)\rho(\Gamma''). \tag{24}$$

Note that the factor

$$\begin{aligned}\int \mathrm{d}\Gamma\, \nu_{\mathcal{O}}(\Gamma; T; X)\nu(\Gamma; T; X) &= \int \mathrm{d}\Gamma[\mathrm{e}^{T\hat{\mathcal{L}}(\Gamma)}\delta(q-X)\mathcal{O}(\Gamma)]\,\mathrm{e}^{T\hat{\mathcal{L}}(\Gamma)}\delta(q-X)\\ &= \int \mathrm{d}\Gamma\, \delta(q-X)\mathcal{O}(\Gamma)\delta(q-X) + \text{b.t.}\\ &= \delta(0)\int \mathrm{d}p\, \mathcal{O}(p, X) + \text{b.t.}\end{aligned} \tag{25}$$

is the overlap between $\nu(\Gamma; 0; X)$ with itself multiplied by $\mathcal{O}(\Gamma)$. Usually, the overlap between squared integrable functions, times another function, can be made one. Then, we can think that this overlap is equivalent to unity for non-normalizable functions. With this in mind, we can think of $\hat{P}_{\mathcal{O}}$ as a projector, in a weak sense, a positive-valued operator if $\mathcal{O}$ is positive. We now have what is necessary for defining a POVM, a positive operator valued measure, for arrival of $\mathcal{O}$ at X in classical systems.

We can recover the full set (for any value of q) of arrival densities by integrating $\hat{P}\rho(\Gamma)$ over X. This is easier to see for $T = 0$:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}X\, \hat{P}\rho(\Gamma) &= \int_{-\infty}^{\infty} \mathrm{d}X\, \nu(\Gamma; 0; X)\int_{-\infty}^{\infty} \mathrm{d}p\, \mathcal{O}(p, X)\rho(p, X)\\ &= \int_{-\infty}^{\infty} \mathrm{d}X\, \delta(q-X)\int_{-\infty}^{\infty} \mathrm{d}p\, \mathcal{O}(p, X)\rho(p, X)\\ &= \int_{-\infty}^{\infty} \mathrm{d}p\, \mathcal{O}(\Gamma)\rho(\Gamma)\\ &= \Pi_{\mathcal{O}}(0, q).\end{aligned} \tag{26}$$

Then, $\int_{-\infty}^{\infty} \mathrm{d}X\, \hat{P}$ projects to the full set of functions $\Pi_{\mathcal{O}}(T, q)$, for any q.

3.5. POV measures

Similar to quantum POV measures [47], classical arrival measures can also be defined in a measurable space (Ω, A) where Ω is a nonempty set and A is a σ-algebra of subsets of Ω. A classical distribution for the arrival of $\mathcal{O}(\Gamma) \geqslant 0$ can be defined as

$$\tau^{\pm}_{\mathcal{O}}(\mathrm{d}T; \rho; X) \equiv \mathrm{d}T\, \Pi^{\pm}_{\mathcal{O}}(T; X). \tag{27}$$

If $\rho(\Gamma)$ is normalized, equation (27) is the 'density' of $\mathcal{O}(\Gamma)$ 'present' at the arrival time. If $\mathcal{O}(\hat{P}, \hat{Q})$ is the velocity, the integral over a time interval is the average number of arrivals (in a given direction) in this time interval. Only if the interval is small, in such a way that one can disregard the multiple crossings, the average number of arrivals is a probability.

4. Traversal time through the Eckart barrier

A brief application of the methods introduced in this paper is the determination of the parts of an initial probability density that arrive before and after the Eckart barrier. This is one of the advantages of using time-front curves.

We have plotted in figure 6 a Gaussian density in phase space with positive momenta and average momentum around the height of the Eckart potential. We have also plotted the time grids for arrival at $X = -3$ and at $X = 3$, i.e. before and after the Eckart potential. Similarly to what happens with free motion, if the arrival point is close to the initial density, the particles that will arrive first are the ones closest to X, in this case the ones with a value of momentum close to the average. If the observation point is far from the initial density, the particles with larger momentum will overcome the ones with smaller momentum and will arrive first. This is what we see in figure 6, the time fronts for arrival at $X = -3$ start touching the initial probability density at the edge with a momentum close to the average. For arrival at the other side of the barrier, the time-front curves begin to intercept the initial density at a point with large momentum, larger as the distance between the arrival point and the initial density increases. This will give the illusion that the particles travel faster over the barrier than a particle with the average momentum. The traversal time for going from $X = -3$ to $X = 3$ is a combination of the traversal times for particles with momentum a bit larger that the average and the times for particles with the largest momentum.

The effect of the barrier is to eliminate part of the density and slow down the particles, in particular the ones with a kinetic energy around the height of the barrier.

5. Remarks

We can say that the support of the probe functions $\nu^{\pm}(\Gamma; T; X)$ indicates the set of points in phase space from which the probability is coming and $\Pi^{\pm}_{\mathcal{O}}(T; X)$ indicates how much $\mathcal{O}$-density is arriving from those points. This is the classical analog of the expression of a quantum wavefunction in terms of time eigenstates. We can call $\nu(\Gamma; T; X)$ a time eigenstate, because its support is made out of points with the same time value.

A characteristic of classical and quantum mechanics is that time enters as a parameter; i.e. once evolution equations are solved, position and momentum are given in terms of time. A goal of this series of papers is to find a picture of these dynamics in which time is no longer a parameter. In particular, in this paper, we have introduced a marginal picture of classical dynamics intended to use the same concepts that were used in quantum mechanics to answer questions regarding the arrival of dynamical quantities at some point $q = X$. In fact this is a picture of classical dynamics which has not previously been fully recognized, a picture in which the representation functions (the probe states) move and the rest, namely probability densities remain static.

In this paper, we have considered motion in one-dimensional systems. For three-dimensional motion, the support of the probe function can be chosen as the propagated surface which start as the plane $q = X$.

The marginal picture for evolution of classical systems introduced in this paper will allow us to treat classical and quantum arrival distributions similarly, but they are marginal functions since the momentum has been summed up. The results in this paper are to be compared with the ones in [40], a paper in which we treat the quantum case. In a forthcoming paper, we will use a non-marginal picture of evolution, one without the summation over p, for the description of classical evolution in an energy–time space.

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